

Integer-Forcing Linear Receivers Based on Lattice Reduction Algorithms

A. Sakzad, *Member, IEEE*, J. Harshan, *Member, IEEE*, and E. Viterbo, *Fellow, IEEE*

Department of Electrical and Computer Systems Engineering,

Monash University, Melbourne, Victoria, Australia

{amin.sakzad, harshan.jagadeesh, and emanuele.viterbo}@monash.edu

Abstract

A new architecture called integer-forcing (IF) linear receiver has been recently proposed for multiple-input multiple-output (MIMO) fading channels, wherein an appropriate integer linear combination of the received symbols has to be computed as a part of the decoding process. Till date, the only known solution to finding these integers is based on exhaustive search. In this paper, we propose a method based on HKZ and Minkowski lattice basis reduction algorithms to obtain the integer coefficients for the IF receiver. We show that the proposed method provides a lower bound on the ergodic rate, and achieves the full receive diversity. Furthermore, we establish the connection between the proposed IF linear receivers and lattice reduction aided MIMO detectors (with equivalent complexity), and point out the advantages of the former class of receivers over the latter. For the 2×2 and 4×4 MIMO channels, we compare the codeword error rate (CER) and bit error rate (BER) of the proposed approach with that of other linear receivers. Simulation results show that the proposed approach outperforms the zero-forcing (ZF) receiver, minimum mean square error (MMSE) receiver, and the lattice reduction aided MIMO detectors.

Index Terms

Lattice reduction algorithm, Minkowski reduction, HKZ reduction, MIMO, linear receivers, integer-forcing.

I. INTRODUCTION

Modern wireless communication systems use multiple antenna transceivers to achieve capacity gains. It is known that such a capacity gain comes at the cost of high decoding complexity at the receiver [20].

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On one extreme, high-complexity joint maximum likelihood (ML) decoders can be used at the receiver to reliably decode the information. On the other extreme, there are well-known *linear receivers* such as the ZF receivers, the MMSE receivers [4], and lattice reduction aided MIMO detectors [12], [17], [18], which reduce the complexity of the decoding process with respect to ML decoding trading off some error performance. The ZF, MMSE, and lattice reduction aided MIMO receivers use the knowledge of the channel state information (CSI) [12], [18] at the receiver. For the ZF and the MMSE receivers, the channel coefficient matrix \mathbf{H} is used *as it is* to recover the information symbols. For the lattice reduction aided MIMO detectors, the matrix \mathbf{H} is reduced to an equivalent channel matrix through a lattice reduction algorithm, prior to recovering the information symbols. The main purpose of this channel reduction is to obtain an equivalent channel matrix which looks more like an orthogonal matrix, which in turn is more suitable for component-wise symbol decoding. It is well known that the lattice reduction aided MIMO decoding algorithms achieve the full receive diversity [12], [13], [17] while the ZF and MMSE receivers only provide receive diversity of one for the $n \times n$ MIMO channel.

A new receiver architecture called integer forcing (IF) linear receiver has been recently proposed in [10], [19], [20] to attain higher rates in MIMO channels with reduced decoding complexity. In this framework, the source employs a layered transmission scheme, and transmits independent codewords simultaneously across the layers. A distinctive property of this scheme is the use of identical lattice codes as codebooks for each layer. At the receiver side, each layer is allowed to decode an integer linear combination of transmitted codewords. Since any integer linear combination of lattice points is another lattice point, the decoded point will be another lattice point, and this substantially reduces the decoding complexity. This idea for MIMO detection was derived from the compute-and-forward approach for physical layer network coding [1], [7], [9], [16], [11]. In the MIMO IF architecture, a filtering matrix \mathbf{B} is used to approximate the channel matrix \mathbf{H} to a “nearest” integer matrix \mathbf{A} [20]. In such a case, finding a non-singular integer matrix \mathbf{A} , whose rows play the role of the coefficients of the linear system of equations, is crucial. Hence, a matrix \mathbf{B} is needed such that \mathbf{A} is full rank, and $\mathbf{B}\mathbf{H} \approx \mathbf{A}$ with minimum quantization error at high signal-to-noise ratio (SNR) values. The problem of finding \mathbf{A} and \mathbf{B} for IF receivers is addressed in [20]. However, its solution is based on an exhaustive search with high computational complexity. In particular, the exhaustive search is computationally complex even for 2×2 real MIMO channel, and becomes impractical to implement for 2×2 complex MIMO and higher order MIMO channels.

In this paper, we propose a practical method for choosing the integer matrix \mathbf{A} . Our method is based on

the Hermite-Korkine-Zolotareff (HKZ) and Minkowski lattice reduction algorithms [3], [8], which were recently developed and employed as part of lattice reduction aided MIMO detectors in [21]. In particular, we use these algorithms to find the matrix \mathbf{A} , which is not only full rank but also unimodular. For the 2×2 and 4×4 MIMO channels, we compare the performance (in terms of ergodic rate and probability of error) of the proposed practical IF solutions with the known linear receivers, and show that (i) it provides a lower bound on the ergodic rate of the IF receiver, (ii) it attains full receive diversity, (iii) it outperforms lattice reduction aided detectors in error performance, and (iv) it trades-off error performance for computational complexity in comparison with the IF receiver based on exhaustive search. We also provide the connection between IF linear receivers and lattice reduction aided MIMO detectors, and point out the benefits of the former class of receivers over the latter.

The rest of the paper is organized as follows. In Section II, we review the background on lattice reduction algorithms. We present the problem statement along with the signal model in Section III. In Section IV, we study the solution to the IF receiver problem based on two lattice reduction algorithms. The definition of the ergodic rate is introduced and a lower bound on the ergodic rate of IF receiver is also presented. In Section V, we point out the differences between the proposed solution to IF receivers and the lattice reduction aided MIMO detectors. In Section VI, we present simulation results on the ergodic rate and the error performance of IF receiver, and compare these results with the traditional linear receivers as well as the lattice reduction aided MIMO detectors. Finally, we present concluding remarks in Section VII.

Notations. Boldface letters are used for row vectors, and capital boldface letters for matrices. We let \mathbb{C} , and $\mathbb{Z}[i]$ denote the field of complex numbers, and the ring of Gaussian integers, respectively, where $i^2 = -1$. We let \mathbf{I}_n and $\mathbf{0}_n$ denote the $n \times n$ identity matrix and zero matrix and the operations $(\cdot)^T$ and $(\cdot)^h$ denote transposition and Hermitian transposition. We let $|\cdot|$ and $\|\cdot\|$ denote the absolute value of a real number and the Euclidean norm of a vector and the operation $\mathbb{E}(\cdot)$ denotes mean of a random variable. We let $\lfloor x \rfloor$ and $\lfloor \mathbf{v} \rfloor$ denote the closest integer to x and the component-wise equivalent operation. We denote the $n \times k$ matrix $\mathbf{X} = [\mathbf{x}_1^T, \dots, \mathbf{x}_k^T]^T$, formed from placing the n -dimensional row vectors $\{\mathbf{x}_m \mid 1 \leq m \leq k\}$ one below each other. The symbol $\mathbf{X}_{j,m}$ denotes the element in the j -th row and m -th column of \mathbf{X} .

II. BACKGROUND ON LATTICES AND LATTICE REDUCTIONS

A d -dimensional *lattice* Λ with a basis set $\{\mathbf{g}_1, \dots, \mathbf{g}_d\} \subseteq \mathbb{R}^d$ is the set of all points of the form $\{\mathbf{x} = \mathbf{u}\mathbf{G} | \mathbf{u} \in \mathbb{Z}^d\}$ where \mathbf{G} is the *generator matrix* of Λ , formed by placing \mathbf{g}_m 's as its rows. The *Gram matrix* of Λ is $\mathbf{M} = \mathbf{G}\mathbf{G}^T$. The m -th *successive minima* of a lattice, denoted by λ_m , is the radius of the smallest possible closed ball around origin containing m or more linearly independent lattice points forming a basis. Given a basis set, a lattice reduction technique is a process to obtain a new basis set of the lattice with shorter vectors. Specifically, for the generator matrix \mathbf{G} , the matrix $\mathbf{G}' = \mathbf{U}\mathbf{G}$ denotes a reduced generator matrix obtained through a lattice reduction technique, where \mathbf{U} is a unimodular matrix. Any matrix \mathbf{G}' can be presented as product of two matrices as $\mathbf{G}' = \mathbf{Q}\mathbf{R}$ where \mathbf{Q} is an orthogonal matrix and \mathbf{R} is an upper triangular matrix.

- A lattice generator matrix \mathbf{G}' is called *Minkowski-reduced* if for $1 \leq m \leq d$, the vectors \mathbf{g}'_m are as short as possible [8]. In particular, \mathbf{G}' is Minkowski-reduced if for $1 \leq m \leq d$, the row vector \mathbf{g}'_m has minimum possible energy amongst all the other lattice points such that $\{\mathbf{g}'_1, \dots, \mathbf{g}'_m\}$ can be extended to another basis of Λ .
- A generator matrix \mathbf{G}' for a lattice Λ is called *HKZ-reduced* [3] if it satisfies
 - 1) $|\mathbf{R}_{m,j}| \leq \frac{1}{2}|\mathbf{R}_{m,m}|$ for all $1 \leq m \leq j \leq d$, and
 - 2) $\mathbf{R}_{j,j}$ be the length of the shortest vector of a lattice generated by the columns of the sub matrix $\mathbf{R}([j, j+1, \dots, d], [j, j+1, \dots, d])$.
- A generator matrix \mathbf{G}' for a lattice Λ is called *LLL-reduced* [6] if it satisfies
 - 1) $|\mathbf{R}_{m,j}| \leq 1/2$ for all $1 \leq m < j \leq d$, and
 - 2) $\delta \|\mathbf{g}'_{m-1}\|^2 \leq \|\mathbf{g}'_m\|^2 + \mathbf{R}_{m,m-1}^2 \|\mathbf{g}'_{m-1}\|^2$ for all $1 < m \leq d$,

where $\delta \in (1/4, 1]$ is a factor selected to achieve a good quality-complexity tradeoff.

For each $1 \leq m \leq d$, it is known that the length of the m -th row vector in \mathbf{G}' is upper bounded by a scaled version of the m -th successive minima of Λ [21]. For the Minkowski reduction, we have

$$\lambda_m^2 \leq \|\mathbf{g}'_m\|^2 \leq \max\left\{1, (5/4)^{d-4}\right\} \lambda_m^2, \text{ for } 1 \leq m \leq d. \quad (1)$$

For the HKZ reduction [5], we have

$$\frac{4\lambda_m^2}{m+3} \leq \|\mathbf{g}'_m\|^2 \leq \frac{(m+3)\lambda_m^2}{4}, \text{ for } 1 \leq m \leq d. \quad (2)$$

For the LLL reduction [6], we have

$$\beta^{1-m}\lambda_m^2 \leq \|\mathbf{g}'_m\|^2 \leq \beta^{d-1}\lambda_m^2, \text{ for } 1 \leq m \leq d, \quad (3)$$

where $\beta = (\delta - 1/4)^{-1}$.

Note that the upper bounds given in (1)–(3) are all scalar multiples of the successive minimas. These scalar multiples are exponential in d for the LLL and the Mikowski reduction algorithms, while polynomial in d for the HKZ reduction algorithm.

III. SIGNAL MODEL AND PROBLEM STATEMENT

We consider a flat-fading MIMO channel with n transmit antennas and n receive antennas. The channel matrix is denoted by $\mathbf{H} \in \mathbb{C}^{n \times n}$, where the entries of \mathbf{H} are i.i.d. as $\mathcal{CN}(0, 1)$. We assume that \mathbf{H} remains fixed for a given interval (of at least N complex channel uses) and take an independent realization in the next interval. We use a n -layer VBLAST transmission scheme where the information transmitted across different antennas are independent. For $1 \leq m \leq n$, the m -th layer is equipped with an encoder $\mathcal{E}_m : \mathcal{R}^k \rightarrow \mathbb{C}^N$ which maps a message $\mathbf{m} \in \mathcal{R}^k$ over the ring \mathcal{R} into a lattice codeword $\mathbf{x}_m \in \Lambda \subset \mathbb{C}^N$ in the complex space. If \mathbf{X} denotes the matrix of transmitted vectors, the received signal \mathbf{Y} is given by

$$\mathbf{Y}_{n \times N} = \sqrt{P}\mathbf{H}_{n \times n}\mathbf{X}_{n \times N} + \mathbf{Z}_{n \times N},$$

where $P = \frac{\text{SNR}}{n}$ and SNR denotes the average signal-to-noise ratio at each receive antenna. We assume that the entries of \mathbf{Z} are i.i.d. as $\mathcal{CN}(0, 1)$. We also assume that \mathbf{H} is known only at the receiver. The goal is to project \mathbf{H} (by left multiplying it with a receiver filtering matrix \mathbf{B}) onto a non-singular integer matrix \mathbf{A} . In order to uniquely recover the information symbols, the matrix \mathbf{A} must be invertible over the ring \mathcal{R} . Thus, we have

$$\mathbf{Y}' = \mathbf{B}\mathbf{Y} = \sqrt{P}\mathbf{B}\mathbf{H}\mathbf{X} + \mathbf{B}\mathbf{Z}. \quad (4)$$

The above signal model is applicable for all linear receivers including the ZF, MMSE (for which $\mathbf{A} = \mathbf{I}$), IF (for which \mathbf{A} is invertible over \mathcal{R}) and lattice reduction aided MIMO detectors (for which \mathbf{A} is unimodular). For the IF receiver [20] formulation, a suitable signal model is

$$\mathbf{Y}' = \sqrt{P}\mathbf{A}\mathbf{X} + \sqrt{P}(\mathbf{B}\mathbf{H} - \mathbf{A})\mathbf{X} + \mathbf{B}\mathbf{Z}, \quad (5)$$

where $\sqrt{P}\mathbf{A}\mathbf{X}$ is the desired signal component, and the effective noise is $\sqrt{P}(\mathbf{B}\mathbf{H} - \mathbf{A})\mathbf{X} + \mathbf{B}\mathbf{Z}$. In particular, the effective noise power along the m -th row of \mathbf{Y}' is defined as

$$g(\mathbf{a}_m, \mathbf{b}_m) \triangleq \|\mathbf{b}_m\|^2 + P\|\mathbf{b}_m\mathbf{H} - \mathbf{a}_m\|^2, \quad (6)$$

where \mathbf{a}_m and \mathbf{b}_m denotes the m -th row of \mathbf{A} and \mathbf{B} , respectively. Note that in order to increase the effective signal-to-noise ratio for each layer, the term $g(\mathbf{a}_m, \mathbf{b}_m)$ has to be minimized for each m by appropriately selecting the matrices \mathbf{A} and \mathbf{B} . We formally put forth the problem statement below:

Problem 1: Given \mathbf{H} and P , the problem is to find the matrices $\mathbf{B} \in \mathbb{C}^{n \times n}$ and $\mathbf{A} \in \mathbb{Z}[i]^{n \times n}$ such that:

- The $\max_{1 \leq m \leq n} g(\mathbf{a}_m, \mathbf{b}_m)$ is minimized, and
- The corresponding matrix \mathbf{A} is invertible over the ring \mathcal{R} .

In [20], the authors considered the invertibility of the matrix \mathbf{A} only over finite fields, in which case is equivalent to requiring $\det(\mathbf{A}) \neq 0$.

A. Known Approaches

If we choose $\mathbf{B} = \mathbf{H}^{-1}$ (or pseudo-inverse of \mathbf{H} in general) and $\mathbf{A} = \mathbf{I}_n$, then we get the ZF receiver. If we choose $\mathbf{B} = \mathbf{H}^h \mathbf{S}^{-1}$ where

$$\mathbf{S} = P^{-1}\mathbf{I}_n + \mathbf{H}\mathbf{H}^h, \quad (7)$$

and $\mathbf{A} = \mathbf{I}_n$, then we get the linear MMSE receiver. With this, the well known linear receivers can be viewed under the umbrella of the IF architecture. However, the ZF and MMSE receivers are known not to minimize $g(\mathbf{a}_m, \mathbf{b}_m)$ [20]. The lattice reduction aided MIMO detector [12] is another approach which will be discussed in details in Section V.

In [20], the authors have proposed a method to solve Problem 1. We now recall the approach presented in [20]. First, conditioned on a fixed $\mathbf{a}_m = \mathbf{a}$, the term $g(\mathbf{a}, \mathbf{b}_m)$ is minimized over all possible values of \mathbf{b}_m . As a result, the optimum value of \mathbf{b}_m can be obtained as

$$\mathbf{b}_m = \mathbf{a}\mathbf{H}^h \mathbf{S}^{-1}. \quad (8)$$

Then, after replacing \mathbf{b}_m of (8) in $g(\mathbf{a}, \mathbf{b}_m)$, the term $g(\mathbf{a}, \mathbf{a}\mathbf{H}^h \mathbf{S}^{-1})$ is minimized over all possible values

of \mathbf{a} to obtain \mathbf{a}_m as $\mathbf{a}_m = \arg \min_{\mathbf{a}} g(\mathbf{a}, \mathbf{a}\mathbf{H}^h\mathbf{S}^{-1})$. The previous expression can be written [20] as

$$\mathbf{a}_m = \arg \min_{\mathbf{a} \in \mathbb{Z}[i]^n} \mathbf{a}\mathbf{V}\mathbf{D}\mathbf{V}^h\mathbf{a}^h, \quad (9)$$

where \mathbf{V} is the matrix composed of the eigenvectors of $\mathbf{H}\mathbf{H}^h$, and \mathbf{D} is a diagonal matrix with m -th entry $\mathbf{D}_{m,m} = (P\rho_m^2 + 1)^{-1}$, where ρ_m is the m -th singular value of \mathbf{H} . With this, we have to obtain n vectors \mathbf{a}_m , $1 \leq m \leq n$, which result in the first n smaller values of $\mathbf{a}\mathbf{V}\mathbf{D}\mathbf{V}^h\mathbf{a}^h$ along with the non-singular property on \mathbf{A} . In order to get \mathbf{a}_m , $1 \leq m \leq n$, the authors of [20] have suggested an exhaustive search for each component of \mathbf{a}_m within a sphere of squared radius

$$1 + P\rho_{\max}^2, \quad (10)$$

where $\rho_{\max} = \max_m \rho_m$. It has also been pointed out in [20] that this search can be accelerated by means of a sphere decoder on the lattice with Gram matrix $\mathbf{M} = \mathbf{V}\mathbf{D}\mathbf{V}^h$. For a fixed P , the complexity of this approach is of order $O(P^n)$. It is also shown in [20] that the exhaustive search approach provides a diversity order of n and a multiplexing gain of n . At this stage, we note that the exhaustive computation of \mathbf{a}_m has high complexity, especially for large values of P and n , and hence the approach in [20] is not practical even for the 2×2 complex case.

IV. PRACTICAL INTEGER-FORCING MIMO RECEIVERS

In this section, we propose a practical method to get an invertible integer matrix \mathbf{A} solving the Problem 1. Once we obtain \mathbf{A} , we compute \mathbf{B} as $\mathbf{B} = \mathbf{A}\mathbf{H}^h\mathbf{S}^{-1}$, where \mathbf{S} is given in (7). Henceforth, we only address method of finding \mathbf{A} .

A. Lattice Reduction Algorithms for IF Architecture

It is pointed out in [20] that the minimization problem in (9) is the shortest vector problem for a lattice with Gram matrix $\mathbf{M} = \mathbf{V}\mathbf{D}\mathbf{V}^h$. Since \mathbf{M} is a positive definite matrix, we can write $\mathbf{M} = \mathbf{L}\mathbf{L}^h$ for some $\mathbf{L} \in \mathbb{C}^{n \times n}$ by using Cholesky decomposition. With this, the rows of $\mathbf{L} = \mathbf{V}\mathbf{D}^{\frac{1}{2}}$ generate a lattice, say Λ . Based on (9), a set of possible choices for $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ is the set of complex integer vectors, whose corresponding lattice points in Λ have lengths at most equal to the n -th successive minima of Λ . The two well-known lattice reduction algorithms satisfying the above property up to constants are HKZ and Minkowski lattice reduction algorithms. In particular, we use HKZ and Minkowski lattice reduction

algorithms, given in [21], to reduce the basis set in \mathbf{L} and obtain a new basis set represented by the rows of \mathbf{L}' . Hence, both the HKZ and the Minkowski reduction algorithm can be employed and the rows of $\mathbf{L}'\mathbf{L}^{-1}$ can be used to obtain the desired matrix \mathbf{A} for IF architecture. Since both \mathbf{L}' and \mathbf{L} are generator matrices for the same lattice Λ , the integer matrix \mathbf{A} is not only invertible over every non-trivial ring but also unimodular. A complex version of LLL lattice reduction algorithm [2] (CLLL) can also be employed. However, since CLLL aims only at obtaining shortest vector in the corresponding basis, it is not a suitable choice for our problem. We summarize the above method as follows:

Input: $\mathbf{H} \in \mathbb{C}^{n \times n}$, and P .

Output: A unimodular matrix \mathbf{A} .

- 1) Form the generator matrix $\mathbf{L} = \mathbf{V}\mathbf{D}^{\frac{1}{2}}$ of a lattice Λ .
- 2) Reduce \mathbf{L} to \mathbf{L}' using either HKZ or Minkowski lattice reduction algorithm.
- 3) The n rows of $\mathbf{L}'\mathbf{L}^{-1}$ provide n rows \mathbf{a}_m of \mathbf{A} for $1 \leq m \leq n$.

For fixed P and n , the expected asymptotic complexity of Minkowski lattice reduction algorithm is upper bounded by $(5/4)^{2n^2}$, while the computational complexity of HKZ lattice reduction is of order $(2\pi e)^{n+O(\log 2n)}$ [21]. Note that unlike the IF receiver based on exhaustive search, the complexity of the lattice reduction techniques are independent of P . Hence, the above algorithms have lower complexity than the exhaustive search and its accelerated version by sphere decoder [15], [20], especially for large values of P . Based on (1) and (2), if $n = 2$ in complex setting (which implies $d = 4$ in (1)), then the Minkowski algorithm provides us a basis set with lengths exactly equal to all successive minimas. For larger values of n , the bounds become loose. The following theorem states that the above mentioned loose bound will not reduce the diversity order of the scheme for large values of n .

Theorem 1: For a MIMO channel with n transmit and n receive antennas over a Rayleigh fading channel, the integer-forcing linear receiver based on lattice reduction achieves full receive diversity.

Proof: The proof is similar to the proof of Theorem 4 in [20] except that equation (197) in [20] becomes an inequality. ■

The complexity and the diversity results of various IF receivers are summarized in Table I. The complexity expressions suggest that HKZ algorithm has lower complexity in comparison with the Minkowski algorithm, especially for higher order MIMO channels.

TABLE I
SUMMARY OF RESULTS.

Name	Approach	Complexity	Receive diversity
Integer-Forcing Linear receiver	Brute Force	P^n	Full
	HKZ reduction	$(2\pi e)^{n+O(\log 2n)}$	Full
	Minkowski reduction	$(5/4)^{2n^2}$	Full

B. Ergodic Rate of IF Linear Receivers

For a given \mathbf{H} matrix, the achievable rate R under the IF architecture is given by [20],

$$R < \min_m nR(\mathbf{H}, \mathbf{a}_m, \mathbf{b}_m), \quad (11)$$

where

$$R(\mathbf{H}, \mathbf{a}_m, \mathbf{b}_m) = \log^+ \left(\frac{P}{g(\mathbf{a}_m, \mathbf{b}_m)} \right), \quad (12)$$

is the achievable rate for the m -th layer of lattice decoding, and $\log^+(x) = \max\{\log(x), 0\}$. Hence, the overall rate is dominated by the layer which has the largest value of $g(\mathbf{a}_m, \mathbf{b}_m)$. Using (12), we now define the ergodic rate of the IF architecture for a $n \times n$ MIMO channel as below.

Definition 2: The ergodic rate R_e of an IF receiver for a MIMO channel is defined as

$$R_e \triangleq \mathbb{E} \left\{ \min_m nR(\mathbf{H}, \mathbf{a}_m, \mathbf{b}_m) \right\},$$

where the mathematical expectation is taken over the channel coefficient matrix \mathbf{H} .

Since \mathbf{a}_m and \mathbf{b}_m are functions of \mathbf{H} , we can alternatively denote $R(\mathbf{H}, \mathbf{a}_m, \mathbf{b}_m)$ as $R(\mathbf{H}, \mathbf{a}_m(\mathbf{H}), \mathbf{b}_m(\mathbf{H}))$. As a result, the definition of the ergodic rate does not depend on a specific matrix pair \mathbf{A} and \mathbf{B} .

We use R'_e to denote the ergodic rate of the IF receiver using the lattice reduction algorithm. The lattice reduction algorithm determines $\bar{\mathbf{a}}_m$ and $\bar{\mathbf{b}}_m$ for a given \mathbf{H} . Thus, the ergodic rate is averaged over various channel gain matrix \mathbf{H} . Based on (12), we get

$$\begin{aligned}
R_e &= \mathbb{E} \left\{ \min_m n \log^+ \left(\frac{P}{g(\mathbf{a}_m, \mathbf{b}_m)} \right) \right\} \\
&\geq \mathbb{E} \left\{ \min_m n \log^+ \left(\frac{P}{g(\bar{\mathbf{a}}_m, \bar{\mathbf{b}}_m)} \right) \right\} \\
&= R'_e.
\end{aligned} \quad (13)$$

where the inequality in (13) holds because \mathbf{a}_m and \mathbf{b}_m obtained using exhaustive search has less or equal energy than $\bar{\mathbf{a}}_m$ and $\bar{\mathbf{b}}_m$ delivered lattice reduction algorithm. Therefore, the ergodic rate of the IF architecture for a MIMO channel is lower bounded by R'_e .

V. COMPARISON OF IF RECEIVERS WITH LATTICE REDUCTION AIDED MIMO DETECTORS

In this section, we point out the differences between the conventional lattice reduction aided MIMO detectors, and the IF receivers based on lattice reduction techniques. In [13], the authors have analyzed two types of lattice reduction aided decoding namely: Type-I and Type-II detectors, and have shown that Type-I method is more appropriate to reduce the effective noise. Hence, we only compare IF receivers with Type-I detectors. In the Type-I technique, the goal of lattice reduction aided MIMO detectors is to reduce \mathbf{H}^{-1} to an equivalent matrix \mathbf{H}' using the lattice reduction algorithms [2], [17], [20]. If the rows of \mathbf{H}' denote the reduced basis set corresponding to the rows of \mathbf{H}^{-1} , then we get $\mathbf{H}' = \mathbf{U}\mathbf{H}^{-1}$, where \mathbf{U} is a unimodular matrix. If we use

$$\mathbf{B} = \mathbf{H}', \quad (14)$$

and $\mathbf{A} = \mathbf{U}$, then (4) can be written as $\mathbf{Y}' = \sqrt{P}\mathbf{U}\mathbf{X} + \mathbf{H}'\mathbf{Z}$. Since \mathbf{U} is a unimodular matrix, it is invertible over \mathcal{R} and the information symbols can be recovered by solving a system of linear equations based on \mathbf{U} . Henceforth, when we use lattice reduction on \mathbf{H}^{-1} and use $\mathbf{B} = \mathbf{U}\mathbf{H}^{-1}$, we refer to such a method as “lattice reduction zero-forcing” (LR-ZF) detector. Apart from the above choice of \mathbf{B} , one can also use

$$\mathbf{B} = \mathbf{U}\mathbf{H}^h\mathbf{S}^{-1}, \quad (15)$$

as in (8) to obtain a better projection matrix. For such a choice, we refer to the method as “lattice reduction MMSE” (LR-MMSE) detector. Note that for large values of P , the matrix $\mathbf{H}^h\mathbf{S}^{-1}$ is the pseudo-inverse of \mathbf{H} , and hence, the LR-ZF detector and the LR-MMSE detector are the same.

We now compare the lattice reduction aided detectors with the IF linear receiver based on lattice reduction techniques. To facilitate the comparison, we study the role of LR-ZF and LR-MMSE detectors in solving the problem addressed in Section III. Along that direction, applying lattice reduction on \mathbf{H}^{-1} can be viewed as the result of substituting $\mathbf{b}_m = \mathbf{a}\mathbf{H}^{-1}$ in the problem of minimizing $g(\mathbf{a}_m, \mathbf{b}_m)$ conditioned on a fixed $\mathbf{a}_m = \mathbf{a}$. However, we already know that the solution to the above conditional minimization problem is given by (8), which is not $\mathbf{b}_m = \mathbf{a}\mathbf{H}^{-1}$. Hence, the choice of \mathbf{B} in (14) and (15) does not

TABLE II
THE ROLE OF IF, LR-ZF, AND LR-MMSE DETECTORS IN SOLVING THE PROBLEM IN SECTION III. IN THIS TABLE, LR DENOTES LATTICE REDUCTION.

	IF receiver	LR-ZF receiver	LR-MMSE receiver
Step 1	Substitute $\mathbf{b}_m = \mathbf{a}\mathbf{H}^h\mathbf{S}^{-1}$ in $g(\mathbf{a}_m, \mathbf{b}_m)$	Substitute $\mathbf{b}_m = \mathbf{a}\mathbf{H}^{-1}$ in $g(\mathbf{a}_m, \mathbf{b}_m)$	Substitute $\mathbf{b}_m = \mathbf{a}\mathbf{H}^{-1}$ in $g(\mathbf{a}_m, \mathbf{b}_m)$
Step 2	$\arg \min_{\mathbf{a} \in \mathbb{Z}[i]^n} \mathbf{a}\mathbf{V}\mathbf{D}\mathbf{V}^h\mathbf{a}^h$	$\arg \min_{\mathbf{a} \in \mathbb{Z}[i]^n} \mathbf{a}\mathbf{H}^{-1}(\mathbf{H}^{-1})^h\mathbf{a}^h$	$\arg \min_{\mathbf{a} \in \mathbb{Z}[i]^n} \mathbf{a}\mathbf{H}^{-1}(\mathbf{H}^{-1})^h\mathbf{a}^h$
Step 3	employ LR on $\mathbf{V}\mathbf{D}^{\frac{1}{2}}$	employ LR on \mathbf{H}^{-1}	employ LR on \mathbf{H}^{-1}
Step 4	use the output of LR as the rows of \mathbf{A}	use the output of LR as the rows of \mathbf{U}	use the output of LR as the rows of \mathbf{U}
Step 5	use $\mathbf{B} = \mathbf{A}\mathbf{H}^h\mathbf{S}^{-1}$	use $\mathbf{B} = \mathbf{U}\mathbf{H}^{-1}$	use $\mathbf{B} = \mathbf{U}\mathbf{H}^h\mathbf{S}^{-1}$

minimize the effective noise. This explains the weakness of the LR-ZF and LR-MMSE receivers in solving the problem posed in Section III, and in-turn explains the benefit of the proposed IF linear receivers. This key difference between the IF receivers and lattice reduction aided detectors can be pointed to Step 1 in Table II. This difference will result in performance degradation of lattice reduction aided decoders in comparison with IF receivers at low and moderate values of P . However, for large values of P , the error performance of LR-ZF, LR-MMSE and the IF receiver based on lattice reduction will coincide since the MMSE solution is known to coincide with the ZF solution at high SNR values. The above advantages are applicable for IF receivers based on lattice reduction techniques. Apart from the above discussed advantages, in general, the IF receiver brings in the following advantages: (i) In the IF receiver, if the computations are done over a finite field, the integer matrix \mathbf{A} must be non-singular, however, in the lattice reduction aided detectors, the integer matrix \mathbf{U} is unimodular, which is a stronger condition than the non-singularity property. This relaxation in the constraint will help the IF receivers in selecting a better integer matrix \mathbf{A} , (ii) The other difference, pointed out in [20], comes from the level of operation of the decoder. The well-known lattice reduction aided detectors work at symbol-level by detecting the symbols of $\mathbf{U}\mathbf{X}$ from the received vectors. However, IF receiver can work on either codeword level or the symbol level.

VI. SIMULATION RESULTS

In this section, we present simulation results on the ergodic rate and the probability of error of various linear receivers. For the ergodic rate, we compare the IF receivers based on the exhaustive search and the lattice reduction algorithm on 2×2 MIMO channels. For the probability of error, we compare the following receiver architectures on 2×2 and 4×4 MIMO channels: (i) IF linear receiver with exhaustive

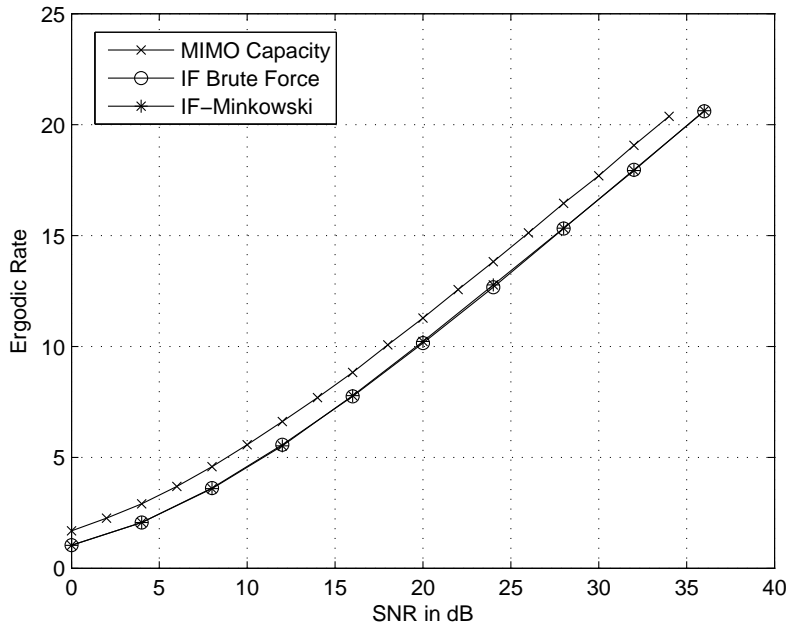


Fig. 1. Ergodic rate of various linear receivers for 2×2 MIMO channel.

search, (ii) IF linear receiver with lattice reduction solutions, (iii) lattice reduction aided detectors, and (iv) the joint maximum likelihood (ML) decoder. For the IF receiver with exhaustive search, the results are presented with the constraint of fixed radius for the exhaustive search. We have not used the radius constraint given in (10) as the corresponding search space increases with P . Instead, we have used a fixed radius of 8 for all values of P . By relaxing this constraint, we have noticeably reduced the complexity of brute force search.

A. Ergodic Rate Results

In Fig. 1, we compare the ergodic rates of the IF receivers with the ergodic MIMO capacity [14] for 2×2 MIMO channel. Note that the IF receiver based on Minkowski lattice reduction provides ergodic rate approximately same as that of the IF receiver based on exhaustive search. For the IF receiver based on exhaustive search, we have searched for a non-singular \mathbf{A} over $\mathbb{Z}[i]$. For higher order MIMO channels, we have not studied the tightness of the lower bound in (13) since the exhaustive search is too complex to implement.

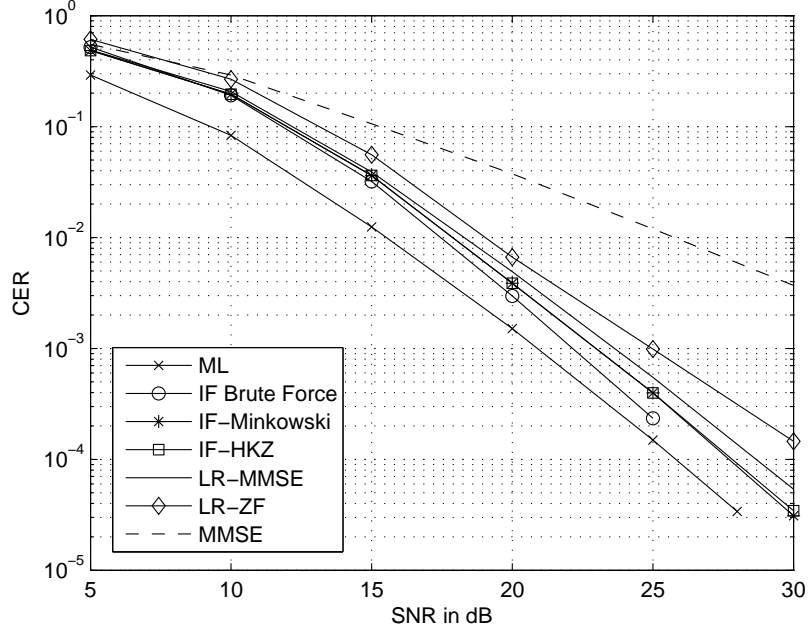


Fig. 2. CER for various IF linear receivers versus lattice reduction aided MIMO detectors with 4-QAM constellation over a 2×2 MIMO channel.

B. Error Probability Results

We now present the CER and BER for various receiver architectures with 4-QAM constellation. We use the finite constellation $\mathcal{S} = \{0, 1, i, 1 + i\}$ carved out of the infinite lattice $\mathbb{Z}[i]$, where $\mathbb{Z}[i] = \mathcal{S} \oplus 2\mathbb{Z}[i]$. In this method, an appropriate translated version of the symbols of \mathcal{S} is transmitted to reduce the average transmit power and removed at the receiver. After suitable modification on the received vector we get, $\mathbf{y} = \sqrt{\frac{\text{SNR}}{n}} \mathbf{H} \mathbf{s} + \mathbf{z}$, where $\mathbf{s} \in \mathcal{S}^{n \times 1}$. Using the standard one-one relation between \mathbb{C} and \mathbb{R}^2 , the received vector \mathbf{y} is unfolded to a real vector [21] to obtain $\bar{\mathbf{y}} = \sqrt{\frac{\text{SNR}}{n}} \bar{\mathbf{H}} \bar{\mathbf{s}} + \bar{\mathbf{z}}$, where $\bar{\mathbf{s}} \in \{0, 1\}^{2n \times 1}$. For this setting, we use modulo lattice decoding at the receiver as follows:

- 1) Each component of $\mathbf{B}\bar{\mathbf{y}}$ is decoded to the nearest point in \mathbb{Z} to get $\hat{\mathbf{y}}$. In particular, we use $\hat{\mathbf{y}} = \lfloor \mathbf{B}\bar{\mathbf{y}} \rfloor$.
- 2) Then, “modulo 2” operation is performed independently on the components of $\hat{\mathbf{y}}$. With this, we get $\mathbf{r} \equiv \hat{\mathbf{y}} \pmod{2}$.
- 3) Further, we solve the system of linear equations $\mathbf{r} \equiv \mathbf{A}\bar{\mathbf{s}} \pmod{2}$ over the ring $\{0, 1\}$ to obtain the decoded vector $\hat{\mathbf{s}}$.

1) *Codeword Error Rate:* We define a codeword error if $\hat{\mathbf{y}} \neq \mathbf{A}\bar{\mathbf{s}}$. In order to obtain the CER results for the IF receiver, we implement the first step of the above decoding procedure. In Fig. 2–3, we present the CER results for all the receiver architectures. For the IF receiver based on exhaustive search, we

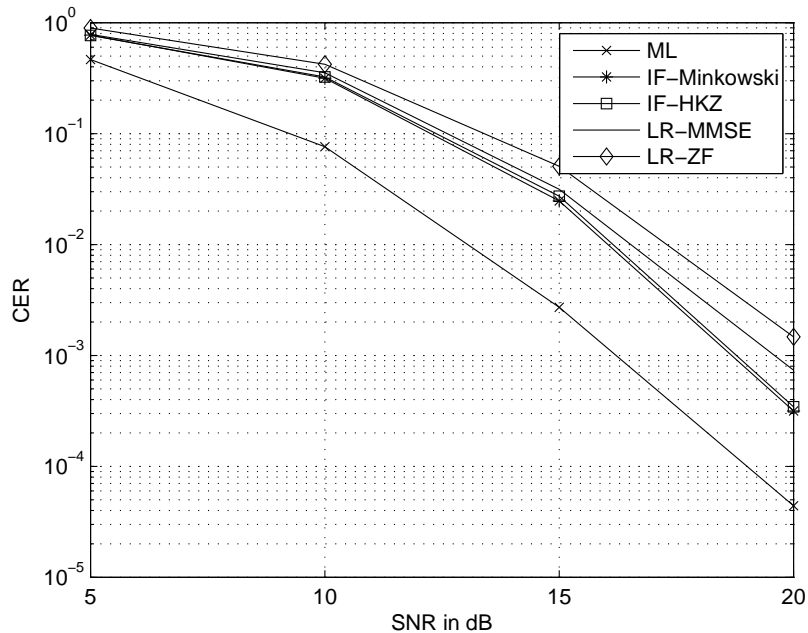


Fig. 3. CER for various IF linear receivers versus lattice reduction aided MIMO detectors with 4-QAM constellation over a 4×4 MIMO channel. The IF Brute Force is computationally complex to simulate in this case.

search for non-singular \mathbf{A} over $\mathbb{Z}[i]$. We refer to such a method as “IF Brute Force”. For the IF receiver based on lattice reduction, the matrix \mathbf{A} is unimodular, and hence is automatically non-singular over $\mathbb{Z}[i]$. From the figures, note that IF receiver with lattice reduction solutions outperform the MMSE, LR-ZF, and LR-MMSE receivers. This difference in the performance between the proposed IF receiver and the lattice reduction aided MIMO detectors confirm the outcome of our comparative study in Section V. For the LR-ZF, and LR-MMSE receivers, we use the Minkowski reduction algorithm, and the decoding operation is same as the IF receiver except that the matrix \mathbf{B} is obtained as in (14) and (15), respectively. Evident from the figures, the proposed lattice reduction solution achieves full receive diversity but trades-off error performance for complexity in comparison with IF brute force search. In the IF receiver based on brute force search, we search for non-singular \mathbf{A} over $\mathbb{Z}[i]$ (but not necessarily invertible over $\mathbb{Z}[i]$, i.e. $\mathbf{A}^{-1} \notin \mathbb{Z}[i]^{n \times n}$), however, in the IF receiver based on lattice reduction, the integer matrix \mathbf{A} is unimodular, which is a stronger condition than the non-singularity property. This relaxation in the constraint can be attributed to the difference in the performance between IF receivers based on exhaustive search and lattice reduction solution. From the figure, it is clear that the ML decoder outperforms the class of linear receivers. For the proposed IF receiver, a lattice reduction algorithm is performed only at the beginning of each quasi-static interval and subsequently, a system of linear equations has to be solved for each codeword use.

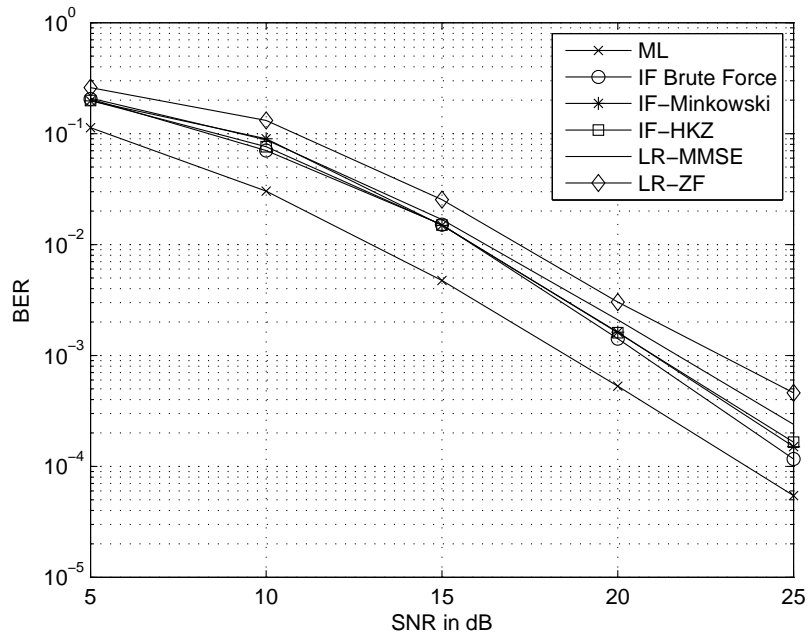


Fig. 4. BER for various IF linear receivers versus lattice reduction aided MIMO detectors with 4-QAM constellation over a 2×2 MIMO channel.

However, for the ML decoder, a sphere decoder algorithm is performed for every codeword use. Hence, the complexity of the proposed IF receiver is lower than the ML decoder for slow varying channels.

2) *Bit Error Rate*: To obtain the BER results of the IF receiver, we have implemented all the steps explained in the decoding procedure. For the IF brute-force search, we have searched for \mathbf{A} which is invertible over \mathcal{S} . This additional constraint is necessary to solve the linear system of equations with a unique solution. It is worthwhile to note that, the matrix \mathbf{A} obtained by our reduction algorithms for IF architecture is unimodular, and hence, it is automatically invertible over \mathcal{S} . In Fig. 4–5, we present the BER results for all the receiver architectures, which shows that the IF receiver with lattice reduction solution marginally trades-off error performance for complexity in comparison with brute force search. In particular, our approach provides diversity results as that of the exhaustive search approach with much lower complexity in comparison with fixed radius exhaustive search. The error performance of lattice reduction aided MIMO detectors are also presented as other methods of low-complexity detectors which achieve full diversity.

VII. CONCLUSIONS

In this paper, we have proposed a systematic method based on HKZ and Minkowski lattice reduction algorithms to obtain integer coefficients for the MIMO IF architecture. This is the first work which provides

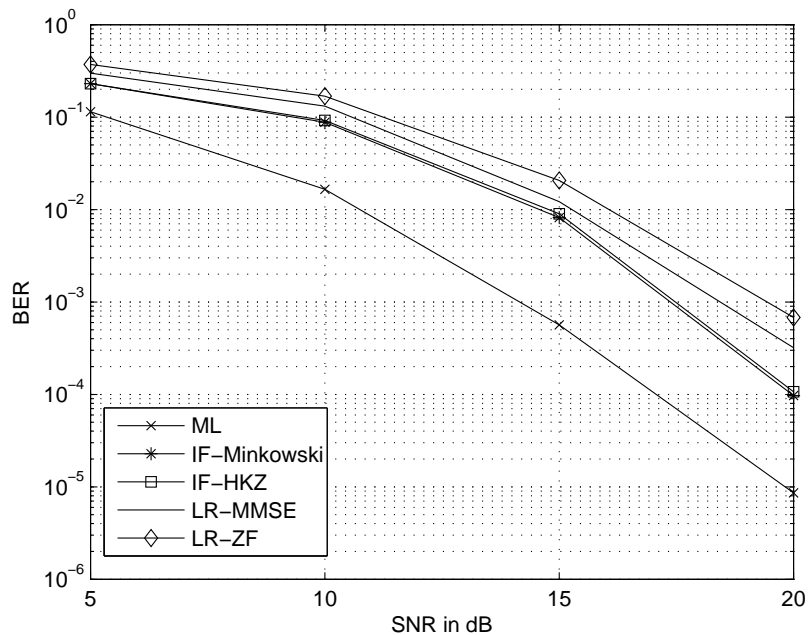


Fig. 5. BER for various IF linear receivers versus lattice reduction aided MIMO detectors with 4-QAM constellation over a 4×4 MIMO channel. The IF Brute Force is computationally complex to simulate in this case.

a practical method to obtain the matrix \mathbf{A} for the IF linear receivers. We have presented the simulation results on the ergodic rate and error performance to reveal the effectiveness of lattice reduction solution in comparison with other linear receivers. We have also shown the connections between our solution and the conventional lattice aided MIMO detectors. In summary, the proposed approach provides full receive diversity at a much lower complexity in comparison with the optimum solution based on exhaustive search.

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